

# The Econometrics of DSGE Models

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EIEF  
Lecture 1: Introduction

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# Outline of the course

- 1 Motivation: DSGE models and their applications
- 2 Approximating and solving DSGE models
  - 1 State space representation
  - 2 Constructing log-linear approximation
  - 3 Higher order approximation techniques
- 3 Frequentist estimation of DSGE models
- 4 Bayesian estimation of DSGE models
- 5 The twilight zone of DSGE estimation

# Course material

## Textbook:

*Canova, Fabio. Methods for applied macroeconomic research. Vol. 13. Princeton University Press, 2011.*

## Papers:

*An, Sungbae, and Frank Schorfheide. "Bayesian analysis of DSGE models." Econometric reviews 26.2-4 (2007): 113-172.*

*More papers will be introduced as we move along*

## Slides:

*Will make them available @ [www.gragusa.org/teaching/eief\\_dsge](http://www.gragusa.org/teaching/eief_dsge)*

# Evaluation

- 1 Final Test (50%)
  - ↳ Take home exam  $\implies$  applied
- 2 Problem sets (50%)
  - ↳ Based on real world example (Julia)
  - ↳ Assigned in class and due in one week

## DSGE quick history

Kydland, Finn E., and Edward C. Prescott. "Time to build and aggregate fluctuations." *Econometrica: Journal of the Econometric Society* (1982): 1345-1370.

- For the first time, macroeconomists had a small and coherent dynamic model of the economy, built from first principles with optimizing agents, rational expectations, and market clearing, that could generate data that resembled observed variables to a remarkable degree.
- There were many dimensions along which the model failed, from the volatility of hours to the persistence of output.
- But the amazing feature was how well the model did despite having so little of what was traditionally thought of as the necessary ingredients of business cycle theories: money, nominal rigidities, or non-market clearing.

## DSGE quick history, ctd.

- The initial reaction to Kydland and Prescott's assertions varied from incredulity to straightforward dismissal
- Reasons? Due to the main characteristics:
  - ▶ The efficiency of business cycles:
    - ★ The bulk of economic fluctuations observed in industrialized countries could be interpreted as an equilibrium outcome resulting from the economy's response to exogenous variations in real forces in an environment with perfect competition and frictionless markets
  - ▶ Importance of technological shocks
    - ★ Basic model able to generate “realistic” fluctuations in output and other macroeconomic variables
  - ▶ The limited role of monetary factors
    - ★ Theory sought to explain economic fluctuations with no reference to monetary factors, even abstracting from the existence of a monetary sector
  - ▶ Technical complexity

## DSGE quick history, ctd.

- Nowadays, DSGE model—leveraging on the initial intuition of Kydland and Prescott's (1982) paper—are flexible enough to accommodate rigidities
  - ▶ **Monopolistic competition:** Without market power, any firm that does not immediately adjust its prices will lose all its sales; (Blanchard and Kiyotaki, 1987)
  - ▶ **Nominal rigidities:** firms are subject to constraints on the frequency with which they can adjust the prices of goods and services they sell (Calvo, 1983)
  - ▶ **Short run non-neutrality of monetary policy:** A monetary policy rule, such as a money growth process or a Taylor rule, is used as a monetary authority.
- Reference:
  - ▶ Woodford, Michael. Interest and prices: Foundations of a theory of monetary policy. Princeton university press, 2011.

# DSGE Characteristics

DSGE models make three modeling choices:

- the behavior of consumers, firms, and financial intermediaries is formally derived from ***microfoundation***;
- the underlying economic environment is that of competitive economy, but with a number of distortions, from nominal rigidities to monopoly power to information problems;
- the model is ***estimated as a system***, rather than equation-by-equation as in the previous generation of models



# DSGE Model

- We will now introduce a simple DSGE model that does not have all the features that I have just enumerated
- This is for two reasons:
  - ▶ I need to keep the presentation of the material accessible—realistic models (Smets and Wouters, 2003) are too large to be used in a didactic setting
  - ▶ Fortunately, we do not lose much. In fact, considering a prototypical DSGE model will help us in highlighting the econometric issues that arise in a realistic settings without having to deal with their complications

# Neoclassical Growth Model

## Setup

$$U = E_0 \sum_{t=1}^{\infty} \beta^{t-1} \frac{(C_t^\lambda (1 - H_t)^{1-\lambda})^{1-\tau}}{1 - \tau}$$

$$Y_t = C_t + I_t$$

$$Y_t = e^{z_t} K_t^\alpha H_t^{1-\alpha}$$

$$K_{t+1} = I_t + (1 - \delta)K_t$$

$$z_t = \rho z_{t-1} + \varepsilon_t, \quad |\rho| < 1$$

$$\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$$

- $C_t$  : consumption,
- $H_t$  : hours
- $Y_t$  : product
- $K_t$  : capital
- $I_t$  : investment
- $z_t$  : technology shocks
- $\varepsilon_t$  : exogenous shock
- $\theta = (\beta, \lambda, \tau, \alpha, \delta, \rho, \sigma_u^2)$   
structural parameters

# Neoclassical Growth Model

- Both welfare theorems hold in this economy.
- Thus, we can solve directly for the social planner's problem:

$$\max_{\{C_t, H_t\}} E_0 \sum_{t=1}^{\infty} \beta^{t-1} \frac{(C_t^\lambda (1-H_t)^{1-\lambda})^{1-\tau}}{1-\tau}$$

subject to

$$C_t + I_t = e^{z_t} K_t^\alpha H_t^{1-\alpha}$$

$$K_{t+1} = I_t + (1-\delta)K_t$$

$$z_t = \rho z_{t-1} + \varepsilon_t, \quad |\rho| < 1, \quad \varepsilon_t \sim N(0, \sigma_\varepsilon^2)$$

$$z_0, K_0$$

- ▶ maximize the utility of the household subject to the production function, the evolution of technology, the law of motion for capital, the resource constraint, and some initial  $k_0$  and  $z_0$ .

# DSGE Models

## Solution

### Policy function

The solution of the dynamic system is a decision rule

$$\underbrace{\zeta_t = r(\zeta_{t-1}, \varepsilon_t; \theta)}_{\text{policy function}}$$

- Policy function are not available in closed form (except for very stylized models)
- We often forced to resort to numerical approximations to characterize the equilibrium dynamics of the model
- Three “historical” approaches
  - ▶ using a linear quadratic approximation of the model (Kydland and Prescott, 1982)
  - ▶ value function iterations (Christiano, 1990)
  - ▶ (log-)linearization of equilibrium conditions (King, Plosser, Rebelo, 2002)

In this course we will focus on the last approach, since it is the one used in conjunction with estimation of the model.

# DSGE Model

## Solution: Linearization

The three main ingredients are:

- 1 Non-stochastic steady state (NSSS)

- 2 Linearization

- 1 e.g.,

$$E[g(x_{t+1})] \approx E[g(\bar{x}) + g'(\bar{x})(x_{t+1} - \bar{x})]$$

- 3 Gensys' canonical form

- 4 Analyze linear rational expectations systems and return solutions for their dependence on exogenous disturbances

- 5 Estimate the parameter vector  $\theta$

- 6 Conduct policy experiments  $\rightarrow$

- 1 what happens if there is a shock to interest rates?

- 2 a shock to technology?

- 3 what is the effect on the macroeconomy if government expenditure are increased?

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# DSGE Models

## Non-stochastic steady state

The NSSS of the model is found by

- 1 setting the shock to  $z_t = \bar{z} = 0$
- 2 dropping the time indexes
- 3 solving for the endogenous variables.
  - 1 This is usually just a matter of getting the algebra right!

# Neoclassical Growth Model

## Linearization

For the Neoclassical Growth Model, the **NSSS** is given:

$$K_{SS} = S_1 / (S_2 S_1 + S_3) \quad (1)$$

$$H_{SS} = S_2 K_{SS} \quad (2)$$

$$C_{SS} = S_3 K_{SS} \quad (3)$$

$$Y_{SS} = K_{SS}^\alpha H_{SS}^{1-\alpha}, \quad (4)$$

where

$$S_1 = \frac{\lambda(1-\alpha)S_2^{-\alpha}}{(1-\lambda)}, \quad S_2 = \frac{(1/\beta - 1 + \delta)^{1/(1-\alpha)}}{\alpha}, \quad S_3 = (S_2^{1-\alpha} - \delta).$$

**Note:** The NSSS is a function of the parameters

$$K_{SS} = K(\theta), \quad H_{SS} = H(\theta), \quad C_{SS} = C(\theta), \quad Y_{SS} = Y(\theta)$$

# DSGE Model

## Linearization

The linearization gives a set of **linear** equations that describe the model around the NSSS.

$$\Gamma_0(\theta)E[x_{t+1}] = \Gamma_1(\theta)x_t + C(\theta) + \Psi(\theta)z_t$$

- The variables ( $x_t$ ) are expressed in **deviation** from the NSSS
- $\Gamma_0(\theta)$ ,  $\Gamma_1(\theta)$ , and  $\Psi(\theta)$  are function of the structural parameter  $\theta$
- Several solution algorithms are available for this linear rational expectations system: e.g.,
  - ▶ Blanchard, Olivier Jean, and Charles M. Kahn. "The solution of linear difference models under rational expectations." *Econometrica: Journal of the Econometric Society* (1980): 1305-1311.
  - ▶ Sims, Christopher A. "Solving linear rational expectations models." *Computational Economics* 20.1 (2002): 1-20.
- All these different linear methods are equivalent, as the linear approximation of a differentiable function is unique and invariant (differentiable parameters transformations)

# Neoclassical Growth Model

## Linearization

For  $k_t = K_t - K_{SS}$  and  $h_t = H_t - H_{SS}$ , the linearized version of the model is

$$Ak_{t+1} + Bk_t + Ch_t + Dz_t = 0 \quad (5)$$

$$E_t(Gk_{t+1} + Hk_t + Jh_{t+1} + Kh_t + Lz_{t+1} + Mz_t) = 0 \quad (6)$$

$$Ez_{t+1} = Nz_t, \quad (7)$$

where the coefficient  $A, B, C, \dots, N$  are **functions** of the structural parameter  $\theta$ .

# Neoclassical Growth Model

Linearization, ctd.

Expression for the coefficient of the policy functions

$$A = 1$$

$$B = (\alpha/K_{SS})C_{SS} - Y_{SS}(\alpha/K_{SS}) - (1 - \delta)$$

$$C = -((\alpha/H_{SS}) + 1/(1 - H_{SS}))C_{SS} - Y_{SS}((1 - \alpha)/H_{SS})$$

$$D = C_{SS} - Y_{SS}$$

$$G = \alpha(\lambda(1 - \tau) - 1)/K_{SS}$$

$$H = (\lambda(1 - \tau) - 1)/K_{SS}$$

$$J = -\alpha(\lambda(1 - \tau) - 1)((\alpha/H_{SS}) + (1/(1 - H_{SS}))) \\ + \beta(\alpha(1 - \alpha)/H_{SS})K_{SS}^{\alpha-1}H_{SS}^{1-\alpha} - ((1 - \tau)(1 - \lambda)/(1 - H_{SS}))$$

$$K = \alpha(\lambda(1 - \tau) - 1)((\alpha/H_{SS}) + (1/(1 - H_{SS}))) - (1 - \lambda)(1 - \tau)/(1 - H_{SS})$$

$$L = \lambda(1 - \tau) - H_{SS} + \alpha\beta K_{SS}^{\alpha-1}H_{SS}^{1-\alpha}$$

$$M = \lambda(1 - \tau) - H_{SS}$$

$$N = \rho$$

# Neoclassical Growth Model

## Solution

The solution of the system of rational expectation equation is given by:

$$k_{t+1} = P(\theta)k_t + Q(\theta)z_t$$

$$h_{t+1} = R(\theta)P(\theta)k_t + (R(\theta)Q(\theta) - S(\theta)\rho)z_t + S(\theta)\varepsilon_{t+1}$$

$$z_{t+1} = \rho z_t + \varepsilon_{t+1}.$$

We can thus write

$$x_{t+1} = G_0(\theta) + G(\theta)x_t + M(\theta)\varepsilon_{t+1}$$

where

$$x_{t+1} = \begin{bmatrix} k_{t+1} \\ h_{t+1} \\ z_{t+1} \end{bmatrix}, \quad G(\theta) = \begin{bmatrix} P(\theta) & 0 & Q(\theta) \\ R(\theta)P(\theta) & 0 & R(\theta)Q(\theta) - S(\theta)\rho \\ 0 & 0 & \rho \end{bmatrix}, \quad M(\theta) = \begin{bmatrix} 0 \\ S(\theta) \\ 1 \end{bmatrix}$$



# Neoclassical Growth Model

Solution, ctd.

The coefficients of the policy functions are:

$$P = \frac{1}{2} \left( \frac{GC}{JA} - \frac{B}{A} - \frac{K}{J} - \sqrt{\left( \frac{\frac{B}{A} + \frac{K}{J} - \frac{GC}{JA}}{JA} \right)^2 - 4 \frac{KB - HC}{JA}} \right)$$

$$R = -\frac{1}{C}(APB)$$

$$Q = \frac{CLN - D(J + N + K)}{AJN + AK - CG - CJR}$$

$$S = \frac{DG + DJR - ALN - AM}{AJN + AK - CG - CJR}$$

# DSGE Models

## 1 Non-stochastic steady state (NSSS)

## 2 Linearization

## 3 Gensys' canonical form

$$\Gamma_0(\theta)E[x_{t+1}] = \Gamma_1(\theta)x_t + C(\theta) + \Psi(\theta)z_t$$

## 4 Solve the system

$$x_{t+1} = G_0(\theta) + G(\theta)x_t + M(\theta)\varepsilon_{t+1}$$

- State/transition equation
- Describe the evolution of the model's **endogenous** variables
- Data???????

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# DSGE Models

## State-space representation

The data enter through the specification of a set of measurement equations:

$$\underbrace{y_t}_{\text{observables}} = H_0(\theta) + H_1(\theta)x_t \underbrace{(+m(\theta)\eta_t)}_{\text{meas. error}}$$

observation equation

## State-space representation

The reduced-form model is given by the following state-space form:

$$x_{t+1} = G_0(\theta) + G(\theta)x_t + M(\theta)\varepsilon_{t+1}$$

$$y_t = H_0(\theta) + H(\theta)x_t (+m(\theta)\eta_{t+1})$$



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# DSGE Models

## Solutions: Remarks

- The “true” model is highly non-linear:
- We end up with a linear model:

$$x_{t+1} = G_0(\theta) + G(\theta)x_t + M(\theta)\varepsilon_{t+1}$$

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- linearization may have important implications
- ...but it tremendously simplifies estimation of the model.

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## Estimation.....

Given the state space model:

$$\begin{aligned}x_{t+1} &= G_0(\theta) + G(\theta)x_t + M(\theta)\varepsilon_{t+1} \\y_t &= H_0(\theta) + H(\theta)x_t (+m(\theta)\eta_t)\end{aligned}$$

and knowing  $\theta$ , the structural parameter vector, we can—using **standard** filtering techniques fit the model to the data and assess its goodness of fit

- This is the approach that we will explore next,
- ...but first let's focus on the “glorious” past

## GMM estimation

- First attempt to estimate parameters of macroeconomic models where limited to subset of  $\theta$
- Consider the Euler conditions of the neoclassical growth model

$$1 = E_t \left[ \beta \left( \frac{C_{t+1}}{C_t} \right)^{\lambda(1-\tau)} \left( \frac{1-H_{t+1}}{1-H_t} \right)^{(1-\lambda)(1-\tau)} R_{t+1} \right]$$

- For simplicity let  $\lambda = 1$ , so that it reduces to

$$1 = E_t \left[ \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\alpha} R_{t+1} \right], \quad \alpha \equiv \tau - 1.$$

The objective is to estimate  $(\beta, \alpha)$ .

- Hansen and Singleton (1982) do that building on Hansen (1982) celebrated GMM paper.

# GMM Estimation

- Notice that the expectation is conditional to the information available at time  $t$
- By the law of iterated expectation

$$E_t \left[ \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\alpha} R_{t+1} - 1 \right] = 0$$

implies

$$E \left[ z_t \left( \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\alpha} R_{t+1} - 1 \right) \right] = 0,$$

for any vector of variables  $z_t$  that is known at time  $t$ .



# GMM Estimation

- Having observation on consumption and interest rates we could estimate  $(\beta, \alpha)$  by solving the empirical moment equation

$$\frac{1}{T} \sum_{t=1}^T z_t \left( \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\alpha} R_{t+1} - 1 \right) = 0.$$

- Two cases:
  - ▶  $dim(z_t) = 2$ : the system is exactly identified and thus is reasonable that a solution exists
  - ▶  $dim(z_t) > 2$ : the system is overidentified, thus no solution exists
- Assume that  $dim(z_t) = m \geq 2$ .

# GMM Estimation

Let

$$h(w_t, \theta) \equiv z_t \left( \beta \left( \frac{C_t}{C_{t-1}} \right)^{-\alpha} R_t - 1 \right),$$

where  $w_{t+1} = \{C_{t+1}, C_t, R_{t+1}\}$  and  $\theta = \{\beta, \alpha\}$ .

The GMM estimator is given by

$$\hat{\theta}^{GMM} = \arg \min_{\theta \in \Theta} \left[ \underbrace{\frac{1}{T} \sum_{t=1}^T h(w_t, \theta)'}_{1 \times m} \right] \underbrace{W}_{m \times m} \left[ \underbrace{\frac{1}{T} \sum_{t=1}^T h(w_t, \theta)}_{m \times 1} \right],$$

where  $W$  is a positive definite matrix.

# GMM Estimation

Let  $\theta_0$  denote the “true” parameters, that is,

$$E[h(w_t, \theta_0)] = 0.$$

Under regularity conditions,

$$\hat{\theta} \xrightarrow{p} \theta_0$$

and

$$\sqrt{T}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, V_W)$$

- The GMM estimator is consistent regardless of the weighting matrix
- The GMM estimator is asymptotically normal with a variance that depends on the weighting matrix  $W$ ,  $V_W$
- Which  $W$  should we use? ...the one that minimizes the asymptotic variance.....

# GMM Estimation

The (optimal, efficient) GMM estimator is defined as

$$\hat{\theta}^{GMM} = \arg \min_{\theta \in \Theta} \left[ \underbrace{\frac{1}{T} \sum_{t=1}^T h(w_t, \theta)'}_{1 \times m} \right] \underbrace{\Omega^{-1}}_{m \times m} \left[ \underbrace{\frac{1}{T} \sum_{t=1}^T h(w_t, \theta)}_{m \times 1} \right],$$

where

$$\Omega = E[g(w_t, \theta_0)g(w_t, \theta_0)'].$$

In this case,

$$\sqrt{T}(\hat{\theta}^{GMM} - \theta) \xrightarrow{d} N(0, V), \quad V = (G' \Omega^{-1} G)^{-1}, \quad G = E[\partial h(w_t, \theta_0) / \partial \theta_0]$$

- Hansen (1982) shows that  $V < V_W$ .

## Two-step procedures

- In practice  $\Omega$  is not known
- Can be estimated using a preliminary consistent estimator of  $\theta_0$ , say  $\tilde{\theta} \xrightarrow{P} \theta_0$

$$\hat{\Omega} = \frac{1}{T} \sum_{t=1}^T h(w_t, \tilde{\theta})h(w_t, \tilde{\theta})'$$

- Then solve

$$\hat{\theta}^{GMM} = \arg \min_{\theta \in \Theta} \left[ \underbrace{\frac{1}{T} \sum_{t=1}^T h(w_t, \theta)'}_{1 \times m} \right] \underbrace{\hat{\Omega}^{-1}}_{m \times m} \left[ \underbrace{\frac{1}{T} \sum_{t=1}^T h(w_t, \theta)}_{m \times 1} \right]$$

- It is a two-step procedure because  $\tilde{\theta}$  is a GMM with non-optimal weighting matrix

## Test of overidentified restrictions

- If the model is correctly specified, that is, there exists a  $\theta_0$  such that

$$E[h(w_t, \theta_0)] = 0,$$

then

$$J = T \left[ \underbrace{\frac{1}{T} \sum_{t=1}^T h(w_t, \hat{\theta}^{GMM})'}_{1 \times m} \right] \underbrace{\hat{\Omega}^{-1}}_{m \times m} \left[ \underbrace{\frac{1}{T} \sum_{t=1}^T h(w_t, \hat{\theta}^{GMM})}_{m \times 1} \right] \xrightarrow{d} \chi_{m-k}^2$$

- $J$  statistics—as it is commonly referred—can then be used to test whether the model is correctly specified (but in reality we are testing whether the **over-identified** restrictions are satisfied)

## Application to US economy

- Per capita real consumption (non-durables plus services)  $C_t$
- Real risk free rate  $r_t^{rf}$ ,  $R_t^{rf} = 1 + r_t^{rf}$
- Instruments

$$z_t^{(1)} = \begin{pmatrix} 1 \\ C_t/C_{t-1} \end{pmatrix}, \quad z_t^{(2)} = \begin{pmatrix} 1 \\ C_t/C_{t-1} \\ 1 + \pi_t \end{pmatrix}$$

## US data

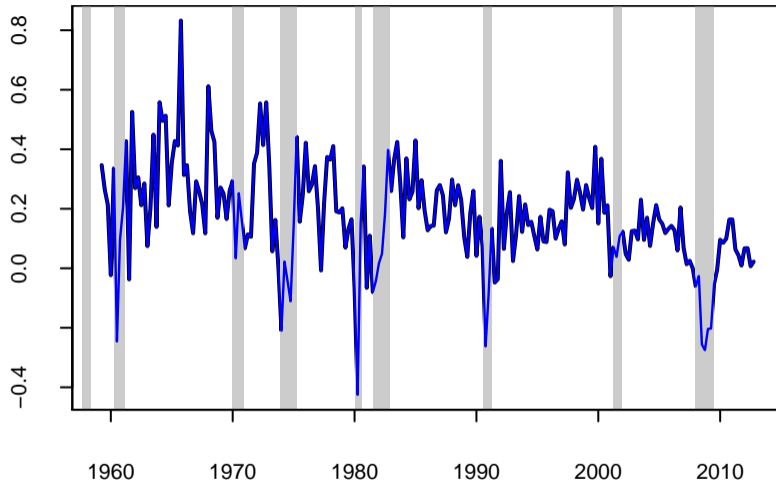


Figure: Per capita real consumption growth, nondurables plus services. Quarterly data: 1959:2-2012:4



## US data

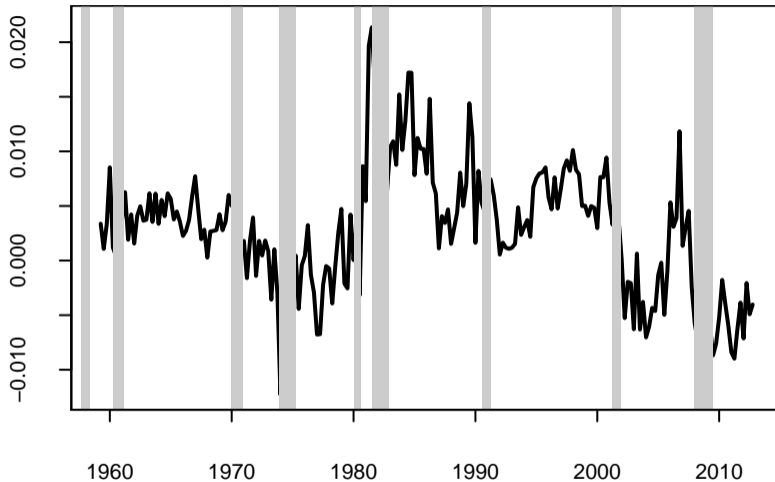


Figure: Real risk free rate. Quarterly data: 1959:2-2012:4

## US data

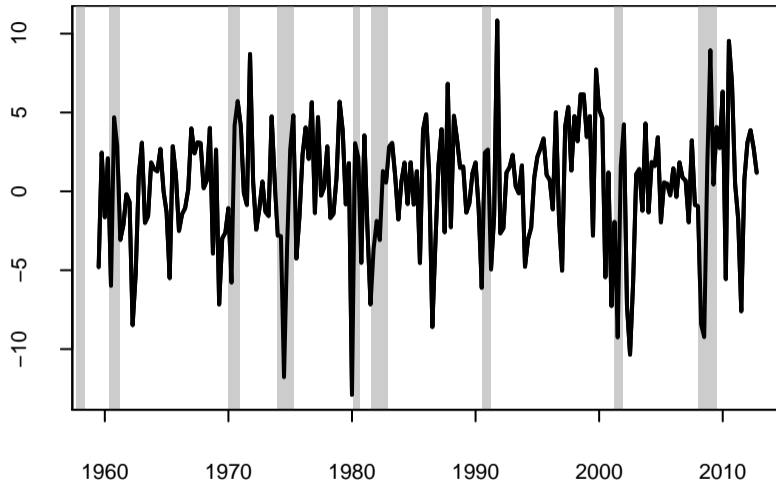


Figure: Excess returns. Quarterly data: 1959:2-2012:4

# GMM estimation: Exact identification

GMM estimation (# iterations: 2)

Coefficients:

	Estimate	Std. Error	t value	Pr(> z )	
theta1	0.8698	0.5238	1.661	0.0484	*
theta2	0.9985	0.0011	899.984	0.0000	***

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1

Objective value: 1.087e-16 on 0 DF, p-value: < 2.2e-16

Observations: 214, moment restrictions: 2

# GMM estimation: Over-identification

GMM estimation (# iterations: 2)

Coefficients:

	Estimate	Std. Error	t value	Pr(> z )	
theta1	0.8795	0.4240	2.074	0.019	*
theta2	0.9990	0.0009	1169.603	0.000	***

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1

Objective value: 25.97 on 1 DF, p-value: 3.46e-07

Observations: 214, moment restrictions: 3

## Testing the model

- The  $J$  statistic is very large:

$$J = 25.97$$

- Since  $m - k = 1$  in this case, the critical value for a 5% significance test is 3.84

$$\Pr(\chi_1^2 > 3.84) \approx 0.05.$$

- Since

$$25.97 > 3.84$$

we reject the null hypothesis the the over-identified conditions hold at the 5% significance level (actually we reject at any reasonable significance level)

# GMM critiques

- Small sample performances of GMM are often very poor
  - ▶ Special issues of Journal of Business and Economic Statistics: vol. 14, no. 3
  - ▶ Hansen, Lars Peter; Heaton, John; Yaron, Yaron (1996). "Finite-sample properties of some alternative GMM estimators". Journal of Business & Economic Statistics 14 (3): 262–280.
- Alternative estimator
  - ▶ Continuous updating GMM
  - ▶ Generalized Empirical Likelihood (GEL)
    - ★ Imbens, Guido W.; Spady, Richard H.; Johnson, Phillip (1998). "Information theoretic approaches to inference in moment condition models". Econometrica 66 (2): 333–357

## Continuous updating

The Continuous Updating GMM estimates minimizes the objective function over  $\theta_0$

$$\arg \min_{\theta \in \Theta} \left[ \underbrace{\frac{1}{T} \sum_{t=1}^T h(w_t, \theta)'}_{1 \times m} \right] \left[ \underbrace{\frac{1}{T} \sum_{t=1}^T h(w_t, \theta) g(w_t, \theta)'}_{m \times m} \right]^{-1} \left[ \underbrace{\frac{1}{T} \sum_{t=1}^T h(w_t, \theta)}_{m \times 1} \right]$$

- In Monte-Carlo experiments this method demonstrated a better performance than the traditional two-step GMM: the estimator has smaller median bias (although fatter tails), and the J-test for overidentifying restrictions in many cases was more reliable
- Implementation is somewhat more difficult.....

# CUE Estimation

MD estimation: "Continuous Updating"

Coefficients:

	Estimate	Std. Error	t value	Pr(> z )
theta1	0.8444	0.3794	2.226	0.013 *
theta2	0.9990	0.0008	1186.192	0.000 ***

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1

Objective value: 29.43 on 1 DF, p-value: 5.801e-08

Observations: 214, moment restrictions: 3, variance: wr



# Alternative moment based estimation procedures

There are other (and somewhat better) techniques to estimate models based on moment conditions

- Empirical Likelihood
- Generalized Empirical Likelihood and Minimum Divergence Estimators (Newey and Smith, 2004; Ragusa, 2013)
- Interestingly they have a Bayesian twist (Ragusa, 2006)

Better means lower bias and small higher order variance