

# The Econometrics of DSGE Models

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# Neoclassical Growth Model

$$U = E_0 \sum_{t=1}^{\infty} \beta^{t-1} \frac{(C_t^\lambda (1-H_t)^{1-\lambda})^{1-\tau}}{1-\tau}$$

$$Y_t = C_t + I_t$$

$$Y_t = e^{z_t} K_t^\alpha H_t^{1-\alpha}$$

$$K_{t+1} = I_t + (1-\delta)K_t$$

$$z_t = \rho z_{t-1} + \varepsilon_t, |\rho| < 1$$

$$\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$$

- $C_t$  : consumption,
- $H_t$  : hours
- $Y_t$  : product
- $K_t$  : capital
- $I_t$  : investment
- $z_t$  : technology shocks
- $u_t$  : exogenous shock
- $\theta = (\beta, \lambda, \tau, \alpha, \delta, \rho, \sigma_u^2)$   
structural parameters

# Neoclassical Growth Model

- Both welfare theorems hold in this economy.
- Thus, we can solve directly for the social planner's problem:

$$\max_{\{C_t, H_t\}} E_0 \sum_{t=1}^{\infty} \beta^{t-1} \frac{(C_t^\lambda (1-H_t)^{1-\lambda})^{1-\tau}}{1-\tau}$$

subject to

$$C_t + I_t = e^{z_t} K_t^\alpha H_t^{1-\alpha}$$

$$K_{t+1} = I_t + (1-\delta)K_t$$

$$z_t = \rho z_{t-1} + \varepsilon_t, \quad |\rho| < 1, \quad \varepsilon_t \sim N(0, \sigma_\varepsilon^2)$$

$$z_0, K_0$$

- ▶ maximize the utility of the household subject to the production function, the evolution of technology, the law of motion for capital, the resource constraint, and some initial  $k_0$  and  $z_0$ .

# Neoclassical Growth Model

## First Order Conditions

The model is **fully** characterized by the first order conditions:

$$1 = E_t \left[ \beta \left( \frac{C_{t+1}}{C_t} \right)^{\lambda(1-\tau)} \left( \frac{1-H_{t+1}}{1-H_t} \right)^{(1-\lambda)(1-\tau)} R_{t+1} \right]$$

$$R_{t+1} \equiv (1-\delta) + e^{z_{t+1}} K_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha}$$

$$(1-\lambda) \frac{1}{(1-H_t)} = \frac{\lambda(1-\alpha) e^{z_t} K_t^\alpha H_t^{-\alpha}}{C_t}$$

$$C_t + K_{t+1} = e^{z_t} K_t^\alpha H_t^{1-\alpha} + (1-\delta)K_t$$

$$z_t = \rho z_{t-1} + \varepsilon_t.$$

# GMM Estimation of DSGE Models

- Only possible for “some” models
- Main problems
  - ▶ shocks are unobservable
  - ▶ some of the variables in the model are not directly observable
- What we would need is GMM when moment restrictions include latent variables
  - ▶ Giacomini, Gallant, Ragusa (2013, 2016)

# GMM Estimation: Example

## Neoclassical Growth Model

Consider the NGM:

$$E_t \left[ \beta \left( \frac{C_{t+1}}{C_t} \right)^{(1-\tau)} \left( \frac{1-H_{t+1}}{1-H_t} \right)^{(1-\lambda)(1-\tau)} R_{t+1} - 1 \right] = 0 \quad (1)$$

$$R_{t+1} \equiv (1-\delta) + e^{z_{t+1}} K_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha}$$
$$(1-\lambda) \frac{1}{(1-H_t)} = \frac{\lambda(1-\alpha)e^{z_t} K_t^\alpha H_t^{-\alpha}}{C_t} \quad (2)$$

$$C_t + K_{t+1} = e^{z_t} K_t^\alpha H_t^{1-\alpha} + (1-\delta)K_t \quad (3)$$

$$z_t = \rho z_{t-1} + \varepsilon_t \quad (4)$$

$$Y_t = e^{z_t} K_t^\alpha H_t^{1-\alpha} \quad (5)$$

There are 7 parameters to estimate  $\theta = (\beta, \lambda, \tau, \alpha, \delta, \rho, \sigma_\varepsilon^2)$ , so we need at least 7 moment conditions.

# GMM Estimation: Example

## Neoclassical Growth Model

We can 3 conditions from the Euler equations

$$E \left\{ \left[ \beta \left( \frac{C_{t+1}}{C_t} \right)^{\lambda(1-\tau)} \left( \frac{1-H_{t+1}}{1-H_t} \right)^{(1-\lambda)(1-\tau)} R_{t+1} - 1 \right] \right\} = 0$$

$$E \left\{ \left[ \beta \left( \frac{C_{t+1}}{C_t} \right)^{\lambda(1-\tau)} \left( \frac{1-H_{t+1}}{1-H_t} \right)^{(1-\lambda)(1-\tau)} R_{t+1} - 1 \right] \frac{C_t}{C_{t-1}} \right\} = 0$$

$$E \left\{ \left[ \beta \left( \frac{C_{t+1}}{C_t} \right)^{\lambda(1-\tau)} \left( \frac{1-H_{t+1}}{1-H_t} \right)^{(1-\lambda)(1-\tau)} R_{t+1} - 1 \right] R_t \right\} = 0$$

Next, we take expectation of the capital equation:

$$0 = E \left[ (1-\lambda) \frac{1}{(1-H_t)} - \frac{\lambda(1-\alpha)e^{z_t} K_t^\alpha H_t^{-\alpha}}{C_t} \right]$$

$$0 = E [C_t + K_{t+1} - e^{z_t} K_t^\alpha H_t^{1-\alpha} - (1-\delta)K_t]$$

# GMM Estimation: Example

## Neoclassical Growth Model

Three moment restrictions are

$$0 = E(z_{t+1} - \rho z_t)$$

$$0 = E[(z_{t+1} - \rho z_t)z_t] = 0$$

$$0 = E[(z_{t+1} - \rho z_t)^2] - \sigma_z^2 = 0$$

In this case we have 8 moment restrictions a 7 parameters to estimate.

$z_t$  is not observable...

However,

$$Y_t = e^{z_t} K_t^\alpha H_t^{1-\alpha} \implies \log Y_t = z_t + \alpha \log K_t + (1 - \alpha) \log H_t,$$

thus

$$z_t = \log Y_t - \alpha \log K_t + (1 - \alpha) \log H_t.$$



## GMM Estimation: Caveat

- Not always possible to obtain expressions for the unobserved shocks, e.g.,

$$E \left\{ \left[ \beta \left( \frac{C_{t+1} v_{t+1}}{C_t v_t} \right)^{(1-\tau)} R_{t+1} - 1 \right] \right\} = 0$$

- All variables need to be observables
  - ▶ We do not have a good measure of capital stock
- Identification difficult to show
- Other technical problems
- Unreasonable estimates of the parameters

# Simulated method of moments

## Basic idea

Suppose we can simulate from the model at a given value of the  $\theta = (\beta, \lambda, \tau, \alpha, \delta, \rho, \sigma_u^2)$ . Let

structural parameters

$$\bar{D}_s(\theta) = \{C_s(\theta), H_s(\theta), K_s(\theta), Y_s(\theta), R_s(\theta)\}, s = 1, \dots, S \rightarrow \infty,$$

the simulated data and  $D_t, t = 1, \dots, T$  the actual data. Then, at the true value of the parameter vector  $\theta_0$ , under regularity conditions,

$$\lim_{S \rightarrow \infty} \frac{1}{S} \sum_{t=1}^S f(\bar{D}_s(\theta_0)) = E[f(D_t)],$$

for a function  $f(\cdot)$ . This suggest the following estimator

$$\min_{\theta \in \Theta} \left[ \frac{1}{T} \sum_{t=1}^T f(D_t) - \frac{1}{S} \sum_{t=1}^S f(\bar{D}_s(\theta)) \right]' W \left[ \frac{1}{T} \sum_{t=1}^T f(D_t) - \frac{1}{S} \sum_{t=1}^S f(\bar{D}_s(\theta)) \right]$$

# Estimation of DSGE Model

## Likelihood approach

Recall the steps to obtaining a *state space* representation a DSGE model

- 1 Obtain first order conditions of the model
- 2 (log) linearize the system of equation, to obtain

$$\Gamma_0(\theta)E[x_{t+1}] = \Gamma_1(\theta)x_t + C(\theta) + \Psi(\theta)z_t$$

- 3 Solve the linear rational expectation system, to obtain

$$x_{t+1} = G_0(\theta) + G(\theta)x_t + M(\theta)\varepsilon_{t+1}$$

- 4 Measurement equation (linking data to model variables)

$$\underbrace{y_t}_{\text{observables}} = H_0(\theta) + H_1(\theta)x_t \underbrace{(+m(\theta)\eta_t)}_{\text{meas. error}}$$

observation equation

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# Linear, gaussian, state space models

Consider the general linear gaussian state space model

$$\alpha_t = c_t + T_t \alpha_{t-1} + R_t \eta_t, \quad \eta_t \sim N(0, Q_t)$$

$$y_t = d_t + Z_t \alpha_t + \varepsilon_t, \quad \varepsilon \sim N(0, H_t)$$

where

$$\alpha_0 \sim N(a_0, \Sigma_0)$$

- The  $p \times 1$  vector  $y_t$  is the observation
- The  $m \times 1$  vector  $\alpha_t$  is the state
- The disturbances  $\varepsilon_t$  and  $\eta_t$  are independent sequences of independent normal vectors.

## Object of interest

Two objects of interest

- 1 **[Kalman Filter]** Filtered state: Estimation of  $E(\alpha_{t+1}|y^t)$  and  $Var(\alpha_{t+1}|y^t)$
- 2 **[State smooter]** Smoothed distribution: Estimation of  $\hat{\alpha}_t = E(\alpha_t|y^T)$  and  $\hat{V}_t = Var(\alpha_t|y^T)$

## Statistical properties, ctd.

- Let

$$\Sigma_t = E(\alpha_t - E(\alpha_t))(\alpha_t - E(\alpha_t))',$$

denote the variance of  $\alpha_t$  at time  $t$

- The variance, by independence of  $\eta_t$  and  $\alpha_t$ , follows recursion

$$\Sigma_{t+1} = T_t \Sigma_t T_t' + R_t Q_t R_t'$$

- If  $T_t$  and  $R_t$  are *constant* and  $T$  is *stable*, the variance converges  $\Sigma$  which solves the Lyapunov equation

$$\Sigma = T \Sigma T' + R Q R'.$$



# The Kalman Filter

The Kalman filter is a *clever* method for computing

- 1  $E(\alpha_t | y^t)$
- 2  $E(\alpha_{t+1} | y^t)$
- 3  $\text{Var}(\alpha_t | y^t)$
- 4  $\text{Var}(\alpha_{t+1} | y^t)$

## Notation

We use the notation

$$\alpha_{t|s} = E(\alpha_t | y^s)$$

$$\Sigma_{t|s} = E(\alpha_t - \alpha_{t|s})(\alpha_t - \alpha_{t|s})'$$

# The Kalman Filter

Suppose we have (from previous recursion)

$$\alpha_{t|t-1} \text{ and } \Sigma_{t|t-1}$$

Two steps:

- 1 [Updating] Yields

$$\alpha_{t|t} \text{ and } \Sigma_{t|t}$$

in terms of  $\alpha_{t|t-1}$  and  $\Sigma_{t|t-1}$ .

- 2 [Prediction] Yields

$$\alpha_{t+1|t} \text{ and } \Sigma_{t+1|t}$$

in terms of  $\alpha_{t|t}$  and  $\Sigma_{t|t}$ .

# The Kalman Filter

Suppose we have (from previous recursion)

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- 2 [Prediction] Yields

$$\alpha_{t+1|t} \text{ and } \Sigma_{t+1|t}$$

in terms of  $\alpha_{t|t}$  and  $\Sigma_{t|t}$ .

# Kalman Filter

The **key result** one needs to keep in mind is the following. If

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \Big| w \sim N \left( \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12} & \Omega_{22} \end{bmatrix} \right)$$

then

$$z_2 | w, z_1 \sim N(m, S)$$

where

$$m = \mu_2 + \Omega_{21} \Omega_{11}^{-1} (z_1 - \mu_1)$$

$$S = \Omega_{22} - \Omega_{21} \Omega_{11}^{-1} \Omega_{12}.$$

## Kalman filter

Suppose we are at  $t = 1$  and  $\alpha_1|y^0 \sim N(\alpha_{1|0}, \Sigma_{1|0})$ . Let  $y^0 = \{\emptyset\}$ .

We have

$$E(y_1|y_0) = d_t + Z_t E(\alpha_1|y^0) + E(\varepsilon_t|y^0) = d_t + Z_t \alpha_{1|0}$$

and

$$y_1 - E(y_1|y_0) = Z_t(\alpha_1 - \alpha_{1|0}) + \varepsilon_t$$

From which it follows that

$$\text{Var}(y_1|y^0) = E[(y_1 - E(y_1|y_0))(y_1 - E(y_1|y_0))'] = Z_t \Sigma_{1|0} Z_t' + H_t$$

$$\begin{aligned} \text{Cov}(y_1, \alpha_1|y^0) &= E[(y_1 - E(y_1|y_0))(\alpha_1 - E(\alpha_1|y_0))'] \\ &= Z_t' E[(\alpha_1 - E(\alpha_1|y_0))(\alpha_1 - E(\alpha_1|y_0))'] \\ &= Z_t' \Sigma_{1|0} \end{aligned}$$

## Kalman filter

Now, since  $\alpha_1|y^0 \sim N(\alpha_{1|0}, \Sigma_{1|0})$  and  $y_1|\alpha_1, y^0$  is also normal, we have

$$\begin{pmatrix} \alpha_1 \\ y_1 \end{pmatrix} \Big| y^0 \sim N \left( \begin{bmatrix} \alpha_{1|0} \\ d_1 + Z_1 \alpha_{1|0} \end{bmatrix}, \begin{bmatrix} \Sigma_{1|0} & \Sigma_{1|0} Z_1 \\ Z_1' \Sigma_{1|0} & Z_1 \Sigma_{1|0} Z_1 + H_1 \end{bmatrix} \right)$$

Now, noting that  $y_1 = (y^0, y_1)$ ,  $\alpha_1|y^1$  can be obtained by the normal formula

$$\alpha_{1|1} = E(\alpha_1|y^1) = \alpha_{1|0} + \Sigma_{1|0} Z_1 (Z_1 \Sigma_{1|0} Z_1 + H_1)^{-1} (y_1 - d_1 - Z_1 \alpha_{1|0})$$

$$\Sigma_{1|1} = \text{Var}(\alpha_1|y^1) = \Sigma_{1|0} - \Sigma_{1|0} Z_1 (Z_1 \Sigma_{1|0} Z_1 + H_1)^{-1} Z_1' \Sigma_{1|0}$$

### Updating

This is the updating step, since we have updated the distribution of  $\alpha_1|y^0$  to  $\alpha_1|y^1$ .

# The Kalman Filter

From

$$\alpha_2 = c_2 + T_2 \alpha_1 + R_2 \eta_2$$

follows that

$$\begin{aligned} \alpha_{2|1} &= c_2 + T_2 \alpha_{1|1} + \underbrace{R_2 \eta_{2|1}}_{= R_2 E[\eta_2 | y^1] = E[\eta_2 | y^1] = 0} \\ &= c_2 + T_2 \alpha_{1|1}, \end{aligned}$$

by independence of  $\eta_t$  and  $y^1$

and similarly

$$\begin{aligned} \Sigma_{2|1} &= E(\alpha_{2|1} - \alpha_2)E(\alpha_{2|1} - \alpha_2)' \\ &= T_2 \Sigma_{1|1} T_2' + R_2 Q_2 R_2' \end{aligned}$$

## Prediction

This is the prediction step

# Kalman filter

We are now at  $t = 2$ , and we know that

$$\alpha_2|y^1 \sim N(c_2 + T_2\alpha_{1|1}, \Sigma_{2|1})$$

thus, we can apply the updating step and the prediction step to obtain the distribution of  $\alpha_3|y^2$  which is given

$$\alpha_3|y^2 \sim N(c_3 + T_3\alpha_{2|2}, T_3\Sigma_{2|2}T_3' + R_3Q_3R_3')$$

where

$$\alpha_{2|2} = \alpha_{2|1} + \Sigma_{2|1}Z_2(Z_2\Sigma_{2|1}Z_2' + H_2)^{-1}(y_2 - d_2 - Z_2\alpha_{2|1})$$

$$\Sigma_{2|2} = \Sigma_{2|1} - \Sigma_{2|1}Z_2(Z_2\Sigma_{2|1}Z_2' + H_2)^{-1}Z_2'\Sigma_{2|1}$$



# The Kalman filter

Thus for a generic  $t$  and given the normality of  $\alpha_t|y^{t-1}$  we can summarize the algorithm as follows

## Updating step

$$\alpha_{t|t} = \alpha_{t|t-1} + \Sigma_{t|t-1} Z_t' (Z_t \Sigma_{t|t-1} Z_t' + H_t)^{-1} (y_t - d_t - Z_t \alpha_{t|t-1})$$
$$\Sigma_{t|t} = \Sigma_{t|t-1} - \Sigma_{t|t-1} Z_t' (Z_t \Sigma_{t|t-1} Z_t' + H_t)^{-1} Z_t \Sigma_{t|t-1},$$

## Prediction step

$$\alpha_{t+1|t} = c_t + T_t \alpha_{t|t}$$
$$\Sigma_{t+1|t} = T_t \Sigma_{t-1|t-1} T_t' + R_t Q_t R_t'$$

# The Kalman filter

## Initial value

- To run the filter we need to *initialize the filter*
  - ▶  $\alpha_0 \sim N(a_0, \Sigma_0)$ ,  $\alpha_1 | Y^0 \sim N(c_1 + T_1 k, R_1 Q_1 R_1' + \Sigma_0)$
  - ▶  $\Sigma_0 = 0$  and  $\alpha_0 = a_0$ , from which  $\alpha_1 | Y^0 \sim N(c_1 + T_1 a_0, R_1 Q_1 R_1')$
- When  $\{c_t, T_t, R_t, Q_t, d_t, Z_t, H_t\}$  are time invariant, common practice is to set  $a_0$  and  $\Sigma_0$  equal to the moment of the unconditional distribution of  $\alpha_t$

$$\alpha_t = c + T\alpha_{t-1} + R\eta_t \implies E(\alpha_t) = (I - T)^{-1}c, \quad \text{vec}(\text{Var}(\alpha_t)) = (I - T \otimes T)^{-1} \text{vec}(Q)$$

provided that the system is stable, that is, the root of

$$\det(I - Tz) = 0$$

lie outside the complex unit circle.

- Other approaches available (we will see them later)
  - ▶ Diffuse Kalman filter (Rosenberg, 1973; Ansley and Kohn, 1985)
  - ▶ Estimate  $a_0$  and  $\Sigma_0$

# The Kalman Filter

## Filtered and smoothed distributions

- Now we now how to recover the following quantities

$$\{\alpha_{t|t-1}\}_{t=1}^T \text{ and } \{\alpha_{t|t}\}_{t=1}^T$$
$$\{\Sigma_{t|t-1}\}_{t=1}^T \text{ and } \{\Sigma_{t|t}\}_{t=1}^T$$

which gives us

- 1 Filtered distribution

$$\alpha_t | y^{t-1} \sim N(\alpha_{t|t-1}, \Sigma_{t|t-1}),$$

- 2 Smoothed distribution

$$\alpha_t | y^T \sim N(\alpha_{t|T}, \Sigma_{t|T}).$$

We can also get something else.....

# The Kalman Filter

## The likelihood function

- The Kalman Filter also gives the likelihood, that is,

$$p(y^T) = \prod_{t=1}^T p(y_t | y^{t-1})$$

- From the updating step

$$\begin{pmatrix} \alpha_t | y^{t-1} \\ y_t | y^{t-1} \end{pmatrix} \sim N \left[ \begin{pmatrix} c_t + T_t \alpha_{t|t-1} \\ d_t + Z_t \alpha_{t|t-1} \end{pmatrix}, \begin{pmatrix} \Sigma_{t|t-1} & \Sigma_{t|t-1} Z_t' \\ Z_t \Sigma_{t|t-1} & Z_t \Sigma_{t|t-1} Z_t' + H_t \end{pmatrix} \right]$$

Thus the likelihood function  $p(y^T)$  is the product of  $T$  multivariate normal densities:

$$p(y_t | y^{t-1}) = \frac{1}{(2\pi)^{k/2} |\Omega_t|^{1/2}} \exp \left( -\frac{1}{2} (u - \mu_t)' \Omega_t^{-1} (u - \mu_t) \right)$$

with

$$\Omega_t = Z_t \Sigma_{t|t-1} Z_t' + H_t, \quad \mu_t = d_t + Z_t \alpha_{t|t-1}$$

# DSGE Model and Kalman Filter

Let's put back in the parameters

$$\underbrace{y_t}_{\text{observables}} = H_0(\theta) + H_1(\theta)x_t \underbrace{(+m(\theta)\eta_t)}_{\text{meas. error}}$$

observation equation

$$y_t = H_0(\theta) + H_1(\theta)x_t (+m(\theta)\eta_t), \quad \eta_t \sim N(0, R(\theta))$$

$$x_t = G_0(\theta) + G(\theta)x_{t-1} + M(\theta)\varepsilon_t, \quad \varepsilon_t \sim N(0, Q(\theta))$$

- For a given  $\theta$ 
  - ▶ Run the Kalman filter
  - ▶ Calculate the likelihood function

$$p(y^T; \theta) = \prod_{t=1}^T p(y_t | y^{t-1}, \theta),$$

where

$$\mu(\theta) = H_0(\theta) + H_1(\theta)x_{t|t-1}$$

$$\Omega(\theta) = H(\theta)\Sigma_{t|t-1}(\theta)H(\theta)' + m(\theta)R(\theta)m(\theta)'$$

# DSGE Model Estimation

Now we can finally see how we estimate  $\theta$ :

- Maximum Likelihood

$$\max_{\theta \in \Theta} \frac{1}{T} \sum_{t=1}^T \log p(y_t | y^{t-1}, \theta),$$

- Bayesian approach

$$p(\theta | y^T) = \frac{p(y^T; \theta) p(\theta)}{p(y^T)}$$

# DSGE Model Steps

For each  $\theta$

- 1 (Log) linearize the system of equation from FOC, to obtain

$$\Gamma_0(\theta)E[x_{t+1}] = \Gamma_1(\theta)x_t + C(\theta) + \Psi(\theta)z_t$$

- 2 Solve the linear rational expectation system, to obtain

$$x_{t+1} = G_0(\theta) + G(\theta)x_t + M(\theta)\varepsilon_{t+1}$$

- 3 Measurement equation (linking data to model variables)

$$\underbrace{y_t}_{\text{observables}} = H_0(\theta) + H_1(\theta)x_t + \underbrace{m(\theta)\eta_t}_{\text{meas. error}}$$

observation equation

- 4 Run the Kalman filter to obtain

$$\ell(y^T, \theta) = \frac{1}{T} \sum_{t=1}^T \log p(y_t | y^{t-1}, \theta)$$

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$$\ell(y^T, \theta) = \frac{1}{T} \sum_{t=1}^T \log p(y_t | y^{t-1}, \theta)$$

# DSGE Model Steps

For each  $\theta$

- 1 (Log) linearize the system of equation from FOC, to obtain

$$\Gamma_0(\theta)E[x_{t+1}] = \Gamma_1(\theta)x_t + C(\theta) + \Psi(\theta)z_t$$

- 2 Solve the linear rational expectation system, to obtain

$$x_{t+1} = G_0(\theta) + G(\theta)x_t + M(\theta)\varepsilon_{t+1}$$

- 3 Measurement equation (linking data to model variables)

$$\underbrace{y_t}_{\text{observables}} = H_0(\theta) + H_1(\theta)x_t + \underbrace{m(\theta)\eta_t}_{\text{meas. error}}$$

observation equation

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# Maximum Likelihood estimation

$$\hat{\theta}_T \equiv \arg \max_{\theta \in \Theta} \sum_{t=1}^T \log p(y_t | y_{t-1}, \theta)$$

Under **regularity conditions**

$$\hat{\theta}_T \xrightarrow{p} \theta, \text{ and } \sqrt{T}(\hat{\theta}_T - \theta) \xrightarrow{d} N(0, H^{-1} \mathcal{I} H^{-1}),$$

where

$$H = -E \left[ \underbrace{\frac{\partial^2 \log \phi(y_t; \mu(\theta), \Omega(\theta))}{\partial \theta \partial \theta'}}_{\text{Hessian of log-likelihood}} \right]$$

and

$$\mathcal{I} = E \left[ \underbrace{\frac{\partial \log \phi(y_t; \mu(\theta), \Omega(\theta))}{\partial \theta'} \frac{\partial \log \phi(y_t; \mu(\theta), \Omega(\theta))}{\partial \theta'}}_{\text{Fischer information matrix}} \right]$$

# Maximum Likelihood estimation

## Regularity conditions

- **Identification:** The parameter vector  $\theta$  is identifiable if

$$\tilde{\theta} \in \Theta \setminus \theta, \text{ then } p(y^T; \tilde{\theta}) \neq p(y^T; \theta),$$

different value of the parameter must correspond to different values of the likelihood function. This assumption is **difficult** to assess for a DSGE model — that although linear are highly non-linear in the structural parameters

- ▶ *Canova, Fabio, and Luca Sala. "Back to square one: identification issues in DSGE models." Journal of Monetary Economics 56.4 (2009): 431-449.*
- ▶ *Iskrev, Nikolay. "Local identification in DSGE models." Journal of Monetary Economics 57.2 (2010): 189-202.*
- ▶ *Komunjer, Ivana, and Serena Ng. "Dynamic identification of dynamic stochastic general equilibrium models." Econometrica 79.6 (2011): 1995-2032.*

# Maximum Likelihood estimation

Regularity conditions, ctd

- **Model is correctly specified:** if it is not, the MLE estimator is still asymptotically consistent, but for a pseudo-true value

$$\hat{\theta}_T \xrightarrow{P} \theta^{PT}$$

The pseudo true value maximize the Kullback Leibler

- ▶ *White, Halbert. "Maximum likelihood estimation of misspecified models." *Econometrica: Journal of the Econometric Society* (1982): 1-25.*

# Maximum Likelihood estimation

## Misspecified likelihood estimation

- Let  $g(y^T)$  the “true” density of the data. The model is misspecified if

$$g(y^T) \notin \{p(y^T; \theta), \theta \in \Theta\}$$

- In this case, it can be shown that

$$\begin{aligned}\theta^{PT} &= \arg \max_{\theta \in \Theta} \int \log \left( \frac{p(y^T; \theta)}{g(y^T)} \right) g(y^T) \\ &= \arg \min_{\theta \in \Theta} \underbrace{\int \log \left( \frac{g(y^T)}{p(y^T; \theta)} \right) g(y^T)}_{\text{Kullback Leibler "distance"}},\end{aligned}$$

minimize the Kullback–Leibler distance (see White, 1982)

- Remember: “all models are false, but some are useful”

# Maximum Likelihood estimation

## Regularity conditions, ctd

- Identification

$$\theta \neq \theta_0 \implies \log p(y^T, \theta) \neq \log p(y^T, \theta_0)$$

- Compactness of parameter space

$\theta \in \Theta$ , where  $\Theta$  is a compact set

- Continuity of  $p(y^T, \theta)$

$$\Pr \left[ \log p(y^T, \theta) \in C^0(\Theta) \right] = 1$$

- Stochastic Dominance

$$|\log p(y_t | y_{t-1}; \theta)| < D(y_t), \quad \text{all } \theta \in \Theta, t \leq T$$

where

$$\int D(y^T) p(y^T, \theta_0) dy^T < \infty$$